Antisymmetric tensor fields on spheres:
functional determinants and non–local counterterms

E. Elizalde[1], M. Lygren[2]

Center for Advanced Studies CEAB, CSIC, Camí de Santa Bàrbara, 17300 Blanes,

and Department ECM and IFAE, Faculty of Physics, University of Barcelona,

Diagonal 647, 08028 Barcelona, Spain

and

D.V. Vassilevich

Department of Theoretical Physics, St. Petersburg University,

198904 St. Petersburg, Russia

Abstract

The Hodge–de Rham Laplacian on spheres acting on antisymmetric tensor fields is considered. Explicit expressions for the spectrum are derived in a quite direct way, confirming previous results. Associated functional determinants and the heat kernel expansion are evaluated. Using this method, new non–local counterterms in the quantum effective action are obtained, which can be expressed in terms of Betti numbers.

PACS: 02.30+g, 02.40.+m

---

[1]E-mail: eli@zeta.ecm.ub.es

[2]E-mail: lygren@fm.unit.no

I. Introduction

Modern interest in antisymmetric tensor fields is connected with supergravity theories where these fields appear as members of a supermultiplet. Kaluza–Klein compactification of higher dimensional supergravities leads to backgrounds of the form $S^d \times R^4$, where S^d is the d-dimensional sphere. Quantum effects on such backgrounds were considered e.g. in [1, 2] Some general mathematical statements about p-forms —very useful in this context— can be found in the monography [3]. Typically, the action for an antisymmetric tensor field B reads

$$S = \int \sqrt{g} dx F_{ij...k} F^{ij...k}, \quad F = dB, \tag{1}$$

where d denotes external differentiation of forms. The action (1) for the p-form B^p is invariant under gauge transformations

$$B^p \to B^p + dB^{p-1}. \tag{2}$$

All quantum corrections in a theory described by the action (1), including the contribution of ghosts, can be expressed in terms of determinants of the Hodge-de Rham Laplacian

$$\Delta_{\mathrm{HdR}} = -(d^*d + dd^*). \tag{3}$$

The spectrum of Δ_{HdR} is the same for p- and $(d\text{-}p)$-forms. Consequently, it is enough to evaluate the determinant of Δ_{HdR} for $p \leq d/2$. Due to the gauge invariance under the transformation (2) and to the factorization property

$$\det_p(-\Delta_{\mathrm{HdR}}) = \det_{pT}(-\Delta_{\mathrm{HdR}}) \times \det_{(p-1)T}(-\Delta_{\mathrm{HdR}}) \tag{4}$$

—where the subscript T means that the determinant is taken over the space of transversal forms— we can restrict our considerations to transversal p-forms. We will assume in all those expressions, as in (4), that zero modes (harmonic p-forms) are excluded from the determinants.

In Sect. 2 we will obtain the spectrum of the Hodge-de Rham Laplace operator on the unit sphere S^d, for any dimension d, acting on transversal p-forms. In Sect. 3 we will calculate explicitly the determinants for the sphere, and in Sect. 4 the heat kernel coefficients. A complete list will be given up to $d = 7$, a dimension that is important in the compactification of supersymmetric theories. Finally, a discussion on the transversal Laplacian and non-local counterterms will be provided in Sect. 5. It is proven there that the heat kernel expansion for the Hodge-de Rham Laplacian on transversal p-forms contains a

constant term even in the case of odd-dimensional spaces. The new non–local counterterms in the quantum effective action will be expressed in terms of Betti numbers.

II. Spectrum of the Laplace operator

In this section we define the spectrum of the Laplace operator Δ_{HdR} on the unit sphere S^d acting on transversal p-forms. For $p = 0, 1$ this spectrum is well known [4]:

$$
\begin{aligned}
b_l^0 &= -l(l+d-1), \quad D_l^0 = \frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}, \quad l = 0, 1, 2, \ldots \\
b_l^1 &= -l(l+d-1)+2-d, \quad D_l^1 = \frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}, \\
& l = 1, 2, 3, \ldots
\end{aligned}
\tag{5}
$$

where b_l^p denote the eigenvalues and D_l^p their degeneracies. For higher forms the spectrum of the Laplace operator on S^d can be obtained by using standard group theoretical techniques [5]–[7].

For any homogeneous space G/H a field Φ_A belonging to an irreducible representation $D(H)$ can be expanded as [8]

$$
\Phi_A(x) = V^{-\frac{1}{2}} \sum_{n,\zeta,q} \sqrt{\frac{d_n}{d_D}} D_{A\zeta,q}^{(n)}(g_x^{-1}) \phi_{q,\zeta}^{(n)} ,
\tag{6}
$$

where V is the volume of G/H and $d_D = \dim D(H)$. We sum over representations $D^{(n)}$ of G which give $D(H)$ after reduction to H. Here ζ labels the multiple components $D(H)$ in the branching $D^{(n)} \downarrow H$, $d_n = \dim D^{(n)}$. The matrix elements of $D^{(n)}$ have the following orthogonality property

$$
\int_{G/H} dx \sqrt{g} D_{A\zeta,q}^{(n)*}(g_x^{-1}) D_{A\xi,p}^{(n')}(g_x^{-1}) = V d_n^{-1} d_D \delta_{\zeta\xi} \delta_{pq} \delta_{nn'}.
\tag{7}
$$

Consider, for example, the case when $d = 5$: $S^5 = SO(6)/SO(5)$. The representations $D(H) = D(SO(5))$ describing antisymmetric tensors are just antisymmetric tensor powers of the vector representation of $SO(5)$:

$$
\begin{aligned}
p &= 0 \qquad D(SO(5)) = [0, 0] \\
p &= 1 \qquad D(SO(5)) = [1, 0] \\
p &= 2 \qquad D(SO(5)) = [1, 1] \\
p &= 3 \qquad D(SO(5)) = [1, 1]
\end{aligned}
\tag{8}
$$

3

We label the representations by their Dynkin indices in square brackets. Notice that we use here the slightly non-standard definition of the Dynkin indices from the book [9]. These indices are more convenient for the reduction of representations. Owing to duality, the representations in the last two lines of (8) are equivalent.

In order to construct the harmonic expansion (6) one must find all the representations of $SO(6)$ which give the representations (8) after reduction to $SO(5)$. For a given irreducible representation $[q_1, q_2]$ of $SO(5)$ with integer Dynkin indices q_1 and q_2, the representations $[m_1, m_2, m_3]$ of $SO(6)$ containing $[q_1, q_2]$ are defined by the conditions [9]

$$m_1 \geq q_1 \geq m_2 \geq q_2 \geq |m_3|, \tag{9}$$

where all m_A are integers and m_1 and m_2 are non-negative. Any representation satisfying the nonequality (9) contains the single representation $[q_1, q_2]$. Summation over ζ in (6) may be omitted. One can easily find all the representations of $SO(6)$ that are needed

$$
\begin{aligned}
p = 0 \quad & D^{(l)}(SO(6)) \;\; = [l, 0, 0], \;\; l = 0, 1, \ldots \\
p = 1 \quad & D^{(l)}(SO(6)) \;\; = [l, 1, 0], \;\; l = 1, 2, \ldots \\
& \qquad\qquad\qquad = [l, 0, 0], \;\; l = 1, 2, \ldots \\
p = 2, 3 \quad & D^{(l)}(SO(6)) \;\; = [l, 1, 0], \;\; l = 1, 2, \ldots \\
& \qquad\qquad\qquad = [l, 1, 1], \;\; l = 1, 2, \ldots \\
& \qquad\qquad\qquad = [l, 1, -1], \;\; l = 1, 2, \ldots
\end{aligned}
\tag{10}
$$

External differentiation maps transversal $p-1$ forms to longitudinal p-forms. This mapping can be traced back to the corresponding spherical harmonics. Hence, for the transversal forms only the following representations contribute to the harmonic expansion:

$$
\begin{aligned}
p = 0 \quad & D^{(l)}(SO(6)) \;\; = [l, 0, 0], \\
p = 1 \quad & D^{(l)}(SO(6)) \;\; = [l, 1, 0], \\
p = 2 \quad & D^{(l)}(SO(6)) \;\; = [l, 1, 1], \\
& \qquad\qquad\qquad = [l, 1, -1], \\
p = 3 \quad & D^{(l)}(SO(6)) \;\; = [l, 1, 0], \qquad l = 1, 2, 3, \ldots
\end{aligned}
\tag{11}
$$

The scalar mode with $l = 0$ belongs to the kernel of the Laplace operator and should be regarded as an harmonic zero-form.

4

In the space of p-forms there are two main second-order differential operators, namely the Hodge–de Rham Laplacian, $-\Delta_{\text{HdR}} = dd^* + d^*d$, and the ordinary Laplacian, $\Delta = \nabla^i \nabla_i$. On the sphere S^d these two operators differ by a constant, namely

$$-\Delta_{\text{HdR}} = \Delta + p^2 - dp. \tag{12}$$

In any homogeneous space G/H the Laplace operators can be expressed in terms of the quadratic Casimir operators of G and H:

$$\Delta = C_2(G) - C_2(H), \quad -\Delta_{\text{HdR}} = C_2(G). \tag{13}$$

Using the harmonic expansion (11), (8), and standard expressions [9] for the Casimir operators in (13), we obtain the eigenvalues b_l^p of the Laplace operator Δ_{HdR} acting on transversal p-forms on S^5. The corresponding degeneracies D_l^p are equal to the dimensions of the representations of $SO(6)$.

$$
\begin{aligned}
b_l^2 &= -l(l+4) - 4, \quad D_l^2 = \frac{1}{2}l(l+1)(l+3)(l+4), \quad l = 1, 2, ... \\
b_l^3 &= -l(l+4) - 3, \quad D_l^3 = \frac{1}{3}l(l+2)^2(l+4), \quad l = 1, 2, ...
\end{aligned} \tag{14}
$$

For $p = 0, 1$ our result coincides with (5). Other spheres can be dealt with along the same lines. For $d = 3$ we have:

$$b_l^2 = -l(l+2), \quad D_l^2 = (l+1)^2, \quad l = 1, 2, ... \tag{15}$$

For $d = 4$:
$$b_l^2 = -l(l+3) - 2, \quad D_l^2 = \frac{1}{2}l(2l+3)(l+3), \quad l = 1, 2, ... \tag{16}$$

For $d = 6$:

$$b_l^2 = b_l^3 = -l(l+5) - 6, \quad D_l^2 = D_l^3 = \frac{1}{12}l(l+1)(l+4)(l+5)(2l+5), \quad l = 1, 2, 3, ... \tag{17}$$

For $d = 7$:

$$
\begin{aligned}
b_l^2 &= -l(l+6) - 8, \qquad D_l^2 = \frac{1}{24}l(l+1)(l+3)^2(l+5)(l+6), \\
b_l^3 &= -l(l+6) - 9, \qquad D_l^3 = \frac{1}{18}l(l+1)(l+2)(l+4)(l+5)(l+6), \quad l = 1, 2, 3, ... \tag{18}
\end{aligned}
$$

Owing to duality, the spectrum of Δ_{HdR} on transversal 4-forms on S^7 is the same as on longitudinal 3-forms. The latter one coincides with that on transversal 2-forms. Continuing in this way, one can define the spectrum for higher values of p. The remarkable property

5

of equivalence of the spectra for $p = 2$ and $p = 3$ on S^6 holds only for transversal forms. In the above equations we have listed some spectra for $p > d/2$. They are useful in some applications not considered in this paper. For example, they are needed for the computation of the spectrum of the Laplacian on a ball [10, 7, 11].

There is a general function for the eigenvalues and their multiplicities, which have been obtained above for some particular cases. It is the following:

$$
\begin{aligned}
D_l(d, p) &= \frac{(2l + d - 1)\,(l + d - 1)!}{p!\,(d - p - 1)!\,(l - 1)!\,(l + p)(l + d - p - 1)}, \\
b_l(d, p) &= -l(l + d - 1) - p(d - p - 1).
\end{aligned}
\tag{19}
$$

These equations can be obtained by means of lengthy but straightforward calculations repeating step by step the above derivation of the spectrum on S^5. All group theoretical techniques that are needed can be learned from Chapts. 9 and 10 of Ref. [9]. Note that an explicit derivation for the case of the ordinary Laplacian had been carried out, e.g., in Ref. [5], and that previous results already existed in the mathematical literature [12, 13]. Our method is similar to the one in the papers [12, 13]. However, explicit expressions for the eigenvalues and degeneracies can be found in Ref. [5] only, where reduction of the harmonic polynomials from R^{d+1} was used. It is noticeable that a mistake in previous calculations was reported in Ref. [5], what shows that the computation is not trivial at all. It thus seems useful to present an alternative derivation of the spectrum, which turns out to be in complete agreement with [5]. Note that the eigenvalues of the Laplace operator on transversal p-forms are denoted in Ref. [5] by $^{p+1}\lambda_k$, where $k = l - 1 = 0, 1, \ldots$.

III. Calculation of the determinants for the sphere

Here we are going to calculate the determinants corresponding to the Hodge-de Rham Laplacian on spheres of different dimensions, $d = 2, 3, 4, 5, 6, 7$, and for forms of different orders $p = 1, 2, 3, 4$. We shall make use of the formulas (see above)

$$
\det(-\Delta_{HdR})^{(d)} = \det(-\Delta_{HdR})_{p^T}^{(d)} \times \det(-\Delta_{HdR})_{(p-1)^T}^{(d)},
\tag{20}
$$

and employ the definition of determinant through the zeta function of the corresponding operator, that is

$$
\det A = \exp\left(-\zeta_A'(0)\right).
\tag{21}
$$

Owing to the multiplicative property of the determinant —which is obviously fulfilled for the operators we are going to consider (see [14] for a discussion of more general cases)— at the

6

level of the zeta functions the product in (20) transforms into a sum of the corresponding zeta functions (even before taking the derivative). We shall arrange our calculations according to this observation. The general methods employed in [15, 16] will be used (see [17], for more references to these techniques).

Using the general formulas (19) for the spectrum and its degeneracy, one can write the expression of the zeta function corresponding to a p-form in any dimension d $(p \leq (d+1)/2)$, namely

$$
\zeta_{-\Delta_{p^T}^{(d)}}(s) = \sum_{l=1}^{\infty} D_l(d,p)\left[-b_l(d,p)\right]^{-s} \tag{22}
$$
$$
= \frac{1}{p!\,(d-p-1)!} \sum_{l=1}^{\infty} \frac{(2l+d-1)(l+d-1)!}{(l-1)!\,(l+p)(l+d-p-1)}\left[\left(l+\frac{d-1}{2}\right)^2 - \left(p-\frac{d-1}{2}\right)^2\right]^{-s}.
$$

To continue, we notice that the degeneracy is a polynomial in l of order $d-1$, and we expand it in powers of $l + (d-1)/2$:

$$
D_l(p,d) = \sum_{\alpha=0}^{d-1} e_\alpha(d,p)\left(l+\frac{d-1}{2}\right)^\alpha. \tag{23}
$$

Formally, we can write

$$
e_\alpha = \frac{1}{\alpha}\frac{d^\alpha}{dl^\alpha}\,D_l(p,d)\big|_{l=\frac{1-d}{2}}. \tag{24}
$$

The sum over l can be evaluated easily, e.g.

$$
\begin{aligned}
\zeta_{-\Delta_{p^T}^{(d)}}(s) &= \sum_{\alpha=0}^{d-1} e_\alpha(d,p) \sum_{l=1}^{\infty}\left(l+\frac{d-1}{2}\right)^\alpha \left[\left(l+\frac{d-1}{2}\right)^2 - \left(p-\frac{d-1}{2}\right)^2\right]^{-s} \\
&= \sum_{\alpha=0}^{d-1} e_\alpha(d,p) \sum_{l=1}^{\infty}\left(l+\frac{d-1}{2}\right)^{\alpha-2s}\left[1-\frac{(p-\frac{d-1}{2})^2}{(l+\frac{d-1}{2})^2}\right]^{-s} \\
&= \frac{1}{\Gamma(s)}\sum_{\alpha=0}^{d-1} e_\alpha(d,p) \sum_{k=0}^{\infty}\frac{\Gamma(k+s)}{k!}\left(p-\frac{d-1}{2}\right)^{2k}\zeta_H(2s+2k-\alpha,\frac{d+1}{2}),
\end{aligned} \tag{25}
$$

where we have used the binomial expansion. Note that this expansion is absolutely convergent, since $[p-(d-1)/2]^2/[l+(d-1)/2]^2 < 1$ for $p \leq (d+1)/2$. The indetermined number 0^0 when $p = (d-1)/2$ is consistently defined to be one. Here $\zeta_H(s,\nu)$ is the Hurwitz zeta-function. For ν a natural number, this zeta-function can be directly related to the Riemann zeta-function through the formula [15]–[17]

$$
\zeta_H(s,m) = \zeta_R(s) - \sum_{l=1}^{m} l^{-s}. \tag{26}
$$

7

For ν a half-integer, we can correspondingly subtract terms from $\zeta_H(s, 1/2)$, which is again related to the Riemann zeta-function

$$\zeta_H(s, 1/2) = (2^s - 1)\zeta_R(s). \tag{27}$$

For d even, $e_\alpha = 0$ for $\alpha = 0, 2, ..., d-2$, and for d odd, $e_\alpha = 0$ for $\alpha = 1, 3, ..., d-2$. When we differentiate the zeta-function (25) we must distinguish between these two cases. For d odd, we get

$$\zeta'_{-\Delta_{p^T}^{(d)}}(0) = \sum_{a=0}^{\frac{d-1}{2}} e_{2a} \left[2\zeta'_H(-2a, \frac{d+1}{2}) + \sum_{k=1}^{\infty} \frac{\left(p - \frac{d-1}{2}\right)^{2k}}{k} \zeta_H(2k - 2a, \frac{d+1}{2}) \right]. \tag{28}$$

While for d even the expression is a bit more complicated since the Hurwitz zeta-function has a pole when its argument is one. Using the Laurent series expansion

$$\zeta_H(2s + 1, \nu) = \frac{1}{2s} - \Psi(\nu) + \mathcal{O}(s), \tag{29}$$

we obtain, for d even,

$$\begin{aligned}
\zeta'_{-\Delta_{p^T}^{(d)}}(0) &= \sum_{a=0}^{\frac{d}{2}-1} e_{2a+1} \left[2\zeta'_H(-2a-1, \frac{d+1}{2}) + \sum_{k=1}^{a} \frac{\left(p - \frac{d-1}{2}\right)^{2k}}{k} \zeta_H(2k - 2a - 1, \frac{d+1}{2}) \right. \\
&+ \frac{\left(p - \frac{d-1}{2}\right)^{2a+2}}{a+1} \left(\frac{1}{2}\sum_{l=1}^{a} l^{-1} - \Psi(\frac{d+1}{2}) \right) \\
&+ \left. \sum_{a+2}^{\infty} \frac{\left(p - \frac{d-1}{2}\right)^{2k}}{k} \zeta_H(2k - 2a - 1, \frac{d+1}{2}) \right]. \tag{30}
\end{aligned}$$

Continuing in this way and substituting for the derivatives of the Riemann zeta function the values [18]

$$\begin{aligned}
\zeta'(0) &= -\frac{1}{2}\ln(2\pi), \quad \zeta'(-1) = -0.1654211437, \quad \zeta'(-2) = -0.0304484571, \\
\zeta'(-3) &= 0.0053785764, \quad \zeta'(-4) = 0.0079838115, \quad \zeta'(-5) = -0.0005729860, \\
\zeta'(-6) &= -0.0058997591, \ \ldots, \tag{31}
\end{aligned}$$

we have obtained the following numerical results for the determinants:

$$\begin{aligned}
\det(-\Delta_{HdR})_4^{(7)} &= 0.088786, \\
\det(-\Delta_{HdR})_3^{(7)} &= 0.088786, \\
\det(-\Delta_{HdR})_2^{(7)} &= 1.858601,
\end{aligned}$$

$$\det(-\Delta_{HdR})_1^{(7)} = 0.775194,$$

$$\det(-\Delta_{HdR})_3^{(6)} = 7.103758,$$

$$\det(-\Delta_{HdR})_2^{(6)} = 1.726306,$$

$$\det(-\Delta_{HdR})_1^{(6)} = 0.835544,$$

$$\det(-\Delta_{HdR})_3^{(5)} = 11.090330,$$

$$\det(-\Delta_{HdR})_2^{(5)} = 11.090330,$$

$$\det(-\Delta_{HdR})_1^{(5)} = 0.581303,$$

$$\det(-\Delta_{HdR})_2^{(4)} = 0.128002,$$

$$\det(-\Delta_{HdR})_1^{(4)} = 0.621433, \tag{32}$$

$$\det(-\Delta_{HdR})_2^{(3)} = 0.095528,$$

$$\det(-\Delta_{HdR})_1^{(3)} = 0.095528,$$

$$\det(-\Delta_{HdR})_1^{(2)} = 10.210016.$$

IV. Calculation of the heat kernel coefficients

The heat-kernel coefficients B_k are given from the small$-t$ expansion of the heat kernel $K(t)$,

$$K(t) = (4\pi t)^{-\frac{d}{2}} \sum_{k=0,\frac{1}{2},1,\dots} B_k t^k \tag{33}$$

When we consider a manifold without boundaries, as is the case for the sphere, the coefficients with a half-integer k vanish. There is a close connection between the coefficients for an operator and its zeta function. This connection is given by the formulas:

$$\text{Res}\,\zeta(s) = \frac{B_{\frac{d}{2}-s}}{(4\pi)^{\frac{d}{2}}\Gamma(s)}, \tag{34}$$

at $s = \frac{m}{2}, \frac{m-1}{2}, \dots, \frac{1}{2}; -\frac{2l+1}{2}$ for $l = 0, 1, 2..$, and

$$\zeta(-m) = (-1)^m m! \frac{B_{\frac{d}{2}+m}}{(4\pi)^{\frac{d}{2}}}, \tag{35}$$

for $m = 0, 1, 2, \dots$ These formulas constitute a very powerful approach to the determination of the heat kernel coefficients (see, for instance, Refs. [19, 20], and the references therein). Again we consider separately the cases d odd and d even. When d is odd we only have to

calculate the residues, since there are only integer coefficients. The residues matching integer coefficients are located at $s = d/2 - m$, $m = 0, 1, 2....$ When $m \leq (d-1)/2$

$$\text{Res} \, \zeta_{-\Delta_{pT}^{(d)}} \left(\frac{d}{2} - m \right) = \frac{1}{2} \sum_{a=\frac{d-1}{2}-m}^{\frac{d-1}{2}} e_{2a} \frac{\Gamma(\frac{1}{2} + a)}{\Gamma(\frac{d}{2} - m)(\frac{1-d}{2} + m + a)!} \left(p - \frac{d-1}{2} \right)^{1-d+2m+2a}, \quad (36)$$

and when $m > (d-1)/2$

$$\text{Res} \, \zeta_{-\Delta_{pT}^{(d)}} \left(\frac{d}{2} - m \right) = \frac{1}{2} \sum_{a=0}^{\frac{d-1}{2}} e_{2a} \frac{\Gamma(\frac{1}{2} + a)}{\Gamma(\frac{d}{2} - m)(\frac{1-d}{2} + m + a)!} \left(p - \frac{d-1}{2} \right)^{1-d+2m+2a}. \quad (37)$$

For even dimension we must consider the point values at $s = -m$ and the residues at $s = d/2 - l$, $l = 0, 1, ..., d/2 - 1$.

$$\zeta_{-\Delta_{pT}^{(d)}}(-m) = \sum_{a=0}^{\frac{d}{2}-1} e_{2a+1} \left[\sum_{k=0}^{m} \frac{(-1)^k m!}{(m-k)! k!} \left(p - \frac{d-1}{2} \right)^{2k} \zeta_H(2k - 2a - 1 - 2m, \frac{d+1}{2}) \right.$$
$$\left. + \frac{(-1)^m m! a!}{2(1 + a + m)!} \left(p - \frac{d-1}{2} \right)^{2+2a+2m} \right], \quad (38)$$

and we get

$$\text{Res} \, \zeta_{-\Delta_{pT}^{(d)}}(\frac{d}{2} - l) = \frac{1}{2} \sum_{a=\frac{d}{2}-1-l}^{\frac{d}{2}-1} e_{2a+1} \frac{a!}{\Gamma(\frac{d}{2} - l)(1 - \frac{d}{2} + l + a)!} \left(p - \frac{d-1}{2} \right)^{2-d+2a+2l}. \quad (39)$$

As the actual zeta functions are sums of the zeta functions for the transversal field, the heat kernel coefficients are also just sums of the corresponding coefficients for the transversal fields. Using the two equations (34) and (35) we immediately obtain the coefficients from the formulas given above.

For $d = 7$, $p = 4$ and $p = 3$, we have:

$$B_0 = \frac{35 \pi^4}{3},$$
$$B_1 = \frac{-175 \pi^4}{3},$$
$$B_2 = \frac{2009 \pi^4}{18},$$
$$B_3 = \frac{-553 \pi^4}{6},$$
$$B_4 = \frac{159 \pi^4}{8},$$
$$B_5 = \frac{167 \pi^4}{24},$$

$$B_6 = \frac{1289\,\pi^4}{720},$$

$$B_7 = \frac{613\,\pi^4}{1680},$$

$$B_8 = \frac{71\,\pi^4}{1152},$$

$$B_9 = \frac{461\,\pi^4}{51840},$$

$$B_{10} = \frac{271\,\pi^4}{241920}.$$

For $d = 7$, $p = 2$:

$$B_0 = 7\,\pi^4,$$

$$B_1 = -21\,\pi^4,$$

$$B_2 = \frac{133\,\pi^4}{10},$$

$$B_3 = \frac{371\,\pi^4}{30},$$

$$B_4 = \frac{-229\,\pi^4}{40},$$

$$B_5 = \frac{-1213\,\pi^4}{120},$$

$$B_6 = \frac{-2807\,\pi^4}{720},$$

$$B_7 = \frac{6483\,\pi^4}{2800},$$

$$B_8 = \frac{132847\,\pi^4}{28800},$$

$$B_9 = \frac{1050881\,\pi^4}{259200},$$

$$B_{10} = \frac{3147083\,\pi^4}{1209600}.,$$

For $d = 7$, $p = 1$:

$$B_0 = \frac{7\,\pi^4}{3},$$

$$B_1 = \frac{7\,\pi^4}{3},$$

$$B_2 = \frac{-301\,\pi^4}{90},$$

$$B_3 = \frac{-203\,\pi^4}{30},$$

$$B_4 = \frac{-43\,\pi^4}{40},$$

11

$$B_5 = \frac{949\,\pi^4}{120},$$

$$B_6 = \frac{6839\,\pi^4}{720},$$

$$B_7 = \frac{9823\,\pi^4}{8400},$$

$$B_8 = \frac{-282271\,\pi^4}{28800},$$

$$B_9 = \frac{-3734393\,\pi^4}{259200},$$

$$B_{10} = \frac{-3734393\,\pi^4}{403200}.$$

For $d = 6$, $p = 3$:

$$B_0 = \frac{64\,\pi^3}{3},$$

$$B_1 = \frac{-256\,\pi^3}{3},$$

$$B_2 = 128\,\pi^3,$$

$$B_3 = \frac{-75008\,\pi^3}{945},$$

$$B_4 = \frac{1472\,\pi^3}{135},$$

$$B_5 = \frac{256\,\pi^3}{99},$$

$$B_6 = \frac{373376\,\pi^3}{405405},$$

$$B_7 = \frac{40448\,\pi^3}{81081},$$

$$B_8 = \frac{1373248\,\pi^3}{3828825},$$

$$B_9 = \frac{167651072\,\pi^3}{535687425},$$

$$B_{10} = \frac{65263383424\,\pi^3}{206239658625}.$$

For $d = 6$, $p = 2$:

$$B_0 = 16\,\pi^3,$$

$$B_1 = -48\,\pi^3,$$

$$B_2 = \frac{128\,\pi^3}{3},$$

$$B_3 = \frac{176\,\pi^3}{315},$$

$$B_4 = \frac{-48\,\pi^3}{5},$$

$$B_5 = \frac{-1360\,\pi^3}{297},$$

$$B_6 = \frac{-171328\,\pi^3}{405405},$$

$$B_7 = \frac{80096\,\pi^3}{81081},$$

$$B_8 = \frac{37236464\,\pi^3}{34459425},$$

$$B_9 = \frac{52062832\,\pi^3}{59520825},$$

$$B_{10} = \frac{2616564224\,\pi^3}{3618239625}.$$

For $d = 6$, $p = 1$:

$$B_0 = \frac{32\,\pi^3}{5},$$

$$B_1 = 0,$$

$$B_2 = \frac{-128\,\pi^3}{15},$$

$$B_3 = \frac{-1408\,\pi^3}{315},$$

$$B_4 = \frac{352\,\pi^3}{75},$$

$$B_5 = \frac{11008\,\pi^3}{1485},$$

$$B_6 = \frac{5982208\,\pi^3}{2027025},$$

$$B_7 = \frac{-164096\,\pi^3}{57915},$$

$$B_8 = \frac{-952002848\,\pi^3}{172297125},$$

$$B_9 = \frac{-337870336\,\pi^3}{72747675},$$

$$B_{10} = \frac{-8650820224\,\pi^3}{4464061875}.$$

For $d = 5$, $p = 3$ and $p = 2$:

$$B_0 = 10\,\pi^3,$$

$$B_1 = \frac{-80\,\pi^3}{3},$$

$$B_2 = \frac{70\,\pi^3}{3},$$

$$B_3 = \frac{-14\,\pi^3}{3},$$

$$B_4 = \frac{-29\,\pi^3}{18},$$

$$B_5 = \frac{-37\,\pi^3}{90},$$

$$B_6 = \frac{-\pi^3}{12},$$

$$B_7 = \frac{-53\,\pi^3}{3780},$$

$$B_8 = \frac{-61\,\pi^3}{30240},$$

$$B_9 = \frac{-23\,\pi^3}{90720},$$

$$B_{10} = \frac{-11\,\pi^3}{388800}.$$

For $d = 5$, $p = 1$:

$$B_0 = 5\,\pi^3,$$

$$B_1 = \frac{-10\,\pi^3}{3},$$

$$B_2 = \frac{-10\,\pi^3}{3},$$

$$B_3 = \frac{2\,\pi^3}{3},$$

$$B_4 = \frac{35\,\pi^3}{18},$$

$$B_5 = \frac{91\,\pi^3}{90},$$

$$B_6 = \frac{-\pi^3}{12},$$

$$B_7 = \frac{-2101\,\pi^3}{3780},$$

$$B_8 = \frac{-3289\,\pi^3}{6048},$$

$$B_9 = \frac{-32791\,\pi^3}{90720},$$

$$B_{10} = \frac{-104873\,\pi^3}{544320}.$$

For $d = 4$, $p = 2$:

$$B_0 = 16\,\pi^2,$$

$$B_1 = -32\,\pi^2,$$

$$B_2 = \frac{304\,\pi^2}{15},$$

$$B_3 = \frac{-160\,\pi^2}{63},$$

$$B_4 = \frac{-176\,\pi^2}{315},$$

$$B_5 = \frac{-608\,\pi^2}{3465},$$

$$B_6 = \frac{-11104\,\pi^2}{135135},$$

$$B_7 = \frac{-448\,\pi^2}{8775},$$

$$B_8 = \frac{-443504\,\pi^2}{11486475},$$

$$B_9 = \frac{-467045792\,\pi^2}{13749310575},$$

$$B_{10} = \frac{-2327539744\,\pi^2}{68746552875}.$$

For $d = 4$, $p = 1$:

$$B_0 = \frac{32\,\pi^2}{3},$$

$$B_1 = \frac{-32\,\pi^2}{3},$$

$$B_2 = \frac{-32\,\pi^2}{45},$$

$$B_3 = \frac{352\,\pi^2}{189},$$

$$B_4 = \frac{928\,\pi^2}{945},$$

$$B_5 = \frac{1952\,\pi^2}{10395},$$

$$B_6 = \frac{-38848\,\pi^2}{405405},$$

$$B_7 = \frac{-262336\,\pi^2}{2027025},$$

$$B_8 = \frac{-3454688\,\pi^2}{34459425},$$

$$B_9 = \frac{-3024736672\,\pi^2}{41247931725},$$

$$B_{10} = \frac{-12244948288\,\pi^2}{206239658625}.$$

For $d = 3$, $p = 2$ and $p = 1$:

$$B_0 = 6\,\pi^2,$$

$$B_1 = -6\,\pi^2,$$

$$B_2 = \pi^2,$$
$$B_3 = \frac{\pi^2}{3},$$
$$B_4 = \frac{\pi^2}{12},$$
$$B_5 = \frac{\pi^2}{60},$$
$$B_6 = \frac{\pi^2}{360},$$
$$B_7 = \frac{\pi^2}{2520},$$
$$B_8 = \frac{\pi^2}{20160},$$
$$B_9 = \frac{\pi^2}{181440},$$
$$B_{10} = \frac{\pi^2}{1814400}.$$

And for $d = 2$, $p = 1$:

$$B_0 = 8\pi,$$
$$B_1 = \frac{-16\pi}{3},$$
$$B_2 = \frac{8\pi}{15},$$
$$B_3 = \frac{32\pi}{315},$$
$$B_4 = \frac{8\pi}{315},$$
$$B_5 = \frac{32\pi}{3465},$$
$$B_6 = \frac{3056\pi}{675675},$$
$$B_7 = \frac{1856\pi}{675675},$$
$$B_8 = \frac{22664\pi}{11486475},$$
$$B_9 = \frac{4481632\pi}{2749862115},$$
$$B_{10} = \frac{104409808\pi}{68746552875}.$$

V. Transversal Laplacian and non-local counterterms

In this section we demonstrate that the heat kernel expansion for the Hodge-de Rham Laplacian on transversal p-forms contains a constant term even in odd–dimensional spaces.

Let us introduce the following notation:

$$\kappa(t) = \sum_{k=1}^{\infty} \exp(-tk^2). \tag{40}$$

The asymptotic behavior of κ in the limit $t \to 0$ is well known

$$\kappa(t) = \frac{1}{2} \left(\sqrt{\frac{\pi}{t}} - 1 \right), \tag{41}$$

where corrections are exponentially small and can be neglected in the computation of power-low asymptotics.

The transversal heat kernel

$$K(t; p, d) = \sum_{l=1}^{\infty} D_l(p, d) \exp\left[t b_l(p, d)\right] \tag{42}$$

on odd-dimensional spheres can be expressed in terms of κ using the explicit form of D_l and b_l of Sect. 2:

$$
\begin{aligned}
K(t; 0, 3) &= -1 - \kappa'(t), \\
K(t; 1, 3) &= -2 \left[\kappa'(t) + \kappa(t) \right], \\
K(t; 0, 5) &= -1 + \frac{1}{12} e^{4t} \left[\kappa''(t) + \kappa'(t) \right], \\
K(t; 1, 5) &= 1 + \frac{1}{3} e^{t} \left[\kappa''(t) + 4\kappa'(t) \right], \\
K(t; 2, 5) &= \frac{1}{2} \left[\kappa''(t) + 5\kappa'(t) + 4\kappa(t) \right], \\
K(t; 0, 7) &= -1 - \frac{e^{9t}}{360} \left[\kappa'''(t) + 5\kappa''(t) + 4\kappa'(t) \right], \\
K(t; 1, 7) &= 1 - \frac{e^{4t}}{60} \left[\kappa'''(t) + 10\kappa''(t) + 9\kappa'(t) \right], \\
K(t; 2, 7) &= -1 - \frac{e^{t}}{24} \left[\kappa'''(t) + 13\kappa''(t) + 36\kappa'(t) \right], \\
K(t; 3, 7) &= -\frac{1}{18} \left[\kappa'''(t) + 14\kappa''(t) + 41\kappa'(t) + 36\kappa(t) \right],
\end{aligned} \tag{43}
$$

where the prime denotes differentiation with respect to the argument.

Now, we can evaluate the coefficient $a_{d/2}$ before t^0 in the small-t expansion of $K(t; p, d)$. Derivatives of κ do not contribute to this coefficient. One obtains the following remarkable relation:

$$a_{d/2} = (-1)^{p+1}. \tag{44}$$

At first sight this relation contradicts the general theory [21] of the heat kernel, which precludes integer powers of t on odd-dimensional manifolds without boundary. In fact,

17

the relation (44) can be derived from general formulae [21]. Consider an odd-dimensional manifold M without boundary. The space of p-forms Λ^p can be decomposed in a direct sum of eigenspaces of the Hodge-de Rham Laplacian:

$$\Lambda^p = \Lambda^{pT} \oplus \Lambda^{pL} \oplus H^p, \tag{45}$$

where Λ^T and Λ^L are transversal and longitudinal p-forms respectively. H^p denotes the space of harmonic p-forms spanned by zero modes of the Hodge-de Rham Laplacian. The Laplace operator on *all* p-forms satisfies the whole set of requirements in [21] and, hence, the corresponding coefficient in front of t^0 in the heat kernel expansion should vanish. On the other hand, this coefficient is just the sum of the coefficients in front of t^0 for the same operator restricted to the spaces on the right hand side of (45). But this immediately gives:

$$0 = a_{d/2}^p + a_{d/2}^{p-1} + \beta_p, \tag{46}$$

where, as above, $a_{d/2}$ denotes the constant term in the heat kernel expansion for transversal p-forms. Here $\beta_p = \dim H^p$ is the Betti number. In particular, for 0-forms we have

$$a_{d/2}^0 = -\beta_0. \tag{47}$$

The two equations (46) and (47) can be solved giving

$$a_{d/2}^p = \sum_{q=0}^{p} (-1)^{p-q+1} \beta_q. \tag{48}$$

The relation (44) is a particular case of (48).

Consider now the quantum path integral for an antisymmetric tensor field with the action (1). The partition function Z_p can be expressed in terms of the determinants of the Hodge-de Rham Laplacian on transversal forms [22, 23]

$$Z_p = \prod_{q=0}^{p} \det_{qT}(-\Delta_{\text{HdR}})^{-\frac{1}{2}(-1)^{p-q}}. \tag{49}$$

To avoid possible ambiguities in treating the zero modes we suppose that $\beta_p = 0$. The "total" heat kernel for Z_p is just an alternated sum of heat kernels for transversal forms. Since the coefficient multiplying t^0 leads to a logarithmic divergence in the path integral, on an odd-dimensional manifold without boundary we have that this divergence is proportional to

$$\sum_{q=0}^{p} (p - q + 1)(-1)^{p-q+1} \beta_q. \tag{50}$$

Such divergency cannot be cancelled by means of an integral of a local invariant constructed from the Riemann tensor and, hence, it requires a non–local counterterm.

Some topological effects in quantum theories of antisymmetric tensor field were discussed in [23]. These effects are however related to the Gauss–Bonnet term, which can be expressed in function of local densities and vanishes for odd–dimensional spheres.

Acknowledgments

References

[1] P. Candelas and S. Weinberg, Nucl. Phys. **B237** (1984) 397.

[2] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B227** (1983) 252.

[3] P.B. Gilkey, *Invariance theory, the heat equation and the Atiyah–Singer theorem* (Publish or Perish, Delaware, 1984)

[4] M.A. Rubin and C. Ordonez, J. Math. Phys. **26** (1985) 65.

[5] I. Iwasaki and K. Katase, Proc. Japan Acad. Sci. **55A** (1979) 141.

[6] V.D. Lyakhovsky, N.N. Shtykov and D.V. Vassilevich, Lett. Math. Phys. **21** (1991) 89; D.V. Vassilevich and N.N.Shtykov, Yadern. Fiz. **53** (1991) 869; Teor. Mat. Fiz. **90** (1992) 12; D.V. Vassilevich, Lett. Math. Phys. **26** (1992) 147.

[7] D.V. Vassilevich, Phys. Lett. **B348** (1995) 39.

[8] A. Salam and J. Strathdee, Ann. Phys. **141** (1982) 556.

[9] A. Barut and R. Raczka, *Theory of group representations and applications* (PWN, Warszawa, 1977).

[10] D.V. Vassilevich, J. Math. Phys. **36** (1995) 3174.

[11] E. Elizalde, M. Lygren and D. Vassilevich, in preparation.

[12] B.L. Beers and R.S. Millman, Amer. J. Math. **99** (1977) 801.

[13] A. Ikeda and Y. Taniguchi, Osaka J. Math. **15** (1978) 515.

[14] E. Elizalde, *On two complementary approaches aiming at the definition of the determinant of an elliptic partial differential operator*, Trento preprint UTF 359, hep-th/9508167 (1995).

[15] E. Elizalde, J. Math. Phys. **35** (1994) 3308.

[16] E. Elizalde, J. Math. Phys. **35** (1994) 6100.

[17] E. Elizalde, *Ten physical applications of spectral zeta functions* (Springer-Verlag, Berlin, 1995); E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications* (World Sci., Singapore, 1994).

[18] E. Elizalde, Math. Computation **47** (1986) 347.

[19] M. Bordag, E. Elizalde and K. Kirsten, *Heat-kernel coefficients of the Laplace operator on the D-dimensional ball*, J. Math. Phys., to appear.

[20] M. Bordag, B. Geyer, K. Kirsten and E. Elizalde, *Zeta-function determinant of the Laplace operator on the D-dimensional ball*, Commun. Math. Phys., to appear.

[21] P.B. Gilkey, J. Diff. Geom. **10** (1975) 601.

[22] W. Siegel, Phys. Lett. **93B** (1980) 170; E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. **D32** (1980) 301; T.Kimura, Prog. Theor. Phys. **65** (1981) 338; I.A. Batalin and G.A.Vilkovisky, Phys. Lett. **120B** (1983) 166.

[23] M.J. Duff and P. van Nieuwenhuizen, Phys. Lett. **94B** (1980) 179; Y.N. Obukhov, Phys. Lett. **109B** (1982) 195.